Formal Proof of Provable Security by Game-playing in a Proof Assistant

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Outline

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Provable Security Paradigm

- Proofs based on complexity theory
- Reduction to well-studied hard problems (RSA, DL etc.)
- In some form of games, asymptotic

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- Proofs by *Code-based* Game-playing
- "the game-playing approach can lead to proofs that are less error-prone and more easily verifiable, even mechanically verifiable"

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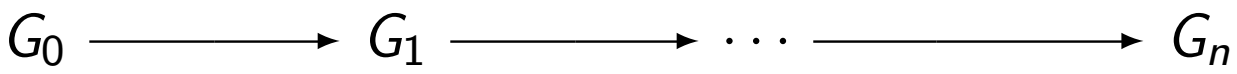
Proofs by *Code-based* Game-playing

Game-playing Proofs are like :



- Sequence of "Games"
- Security definitions are games
- Proof steps are game transformations

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$$G_0 \longrightarrow G_1 \longrightarrow \cdots \longrightarrow G_n$$

P_0

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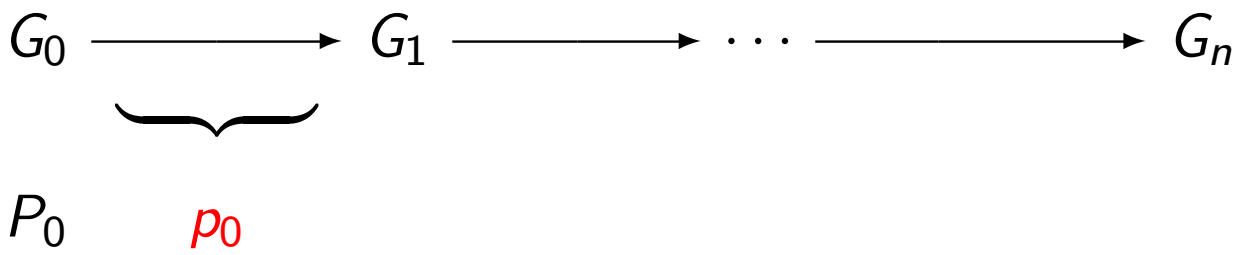

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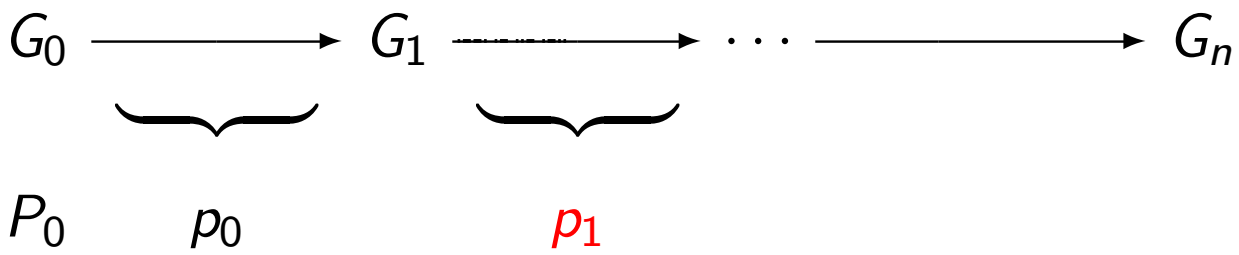

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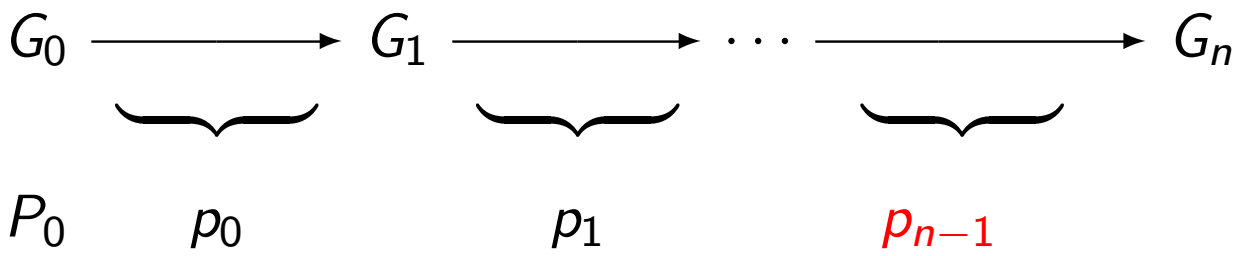

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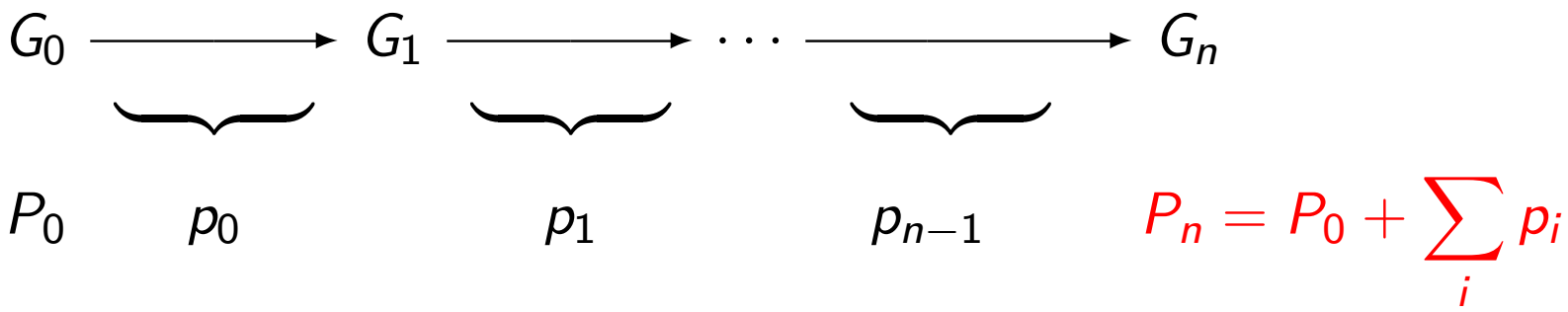

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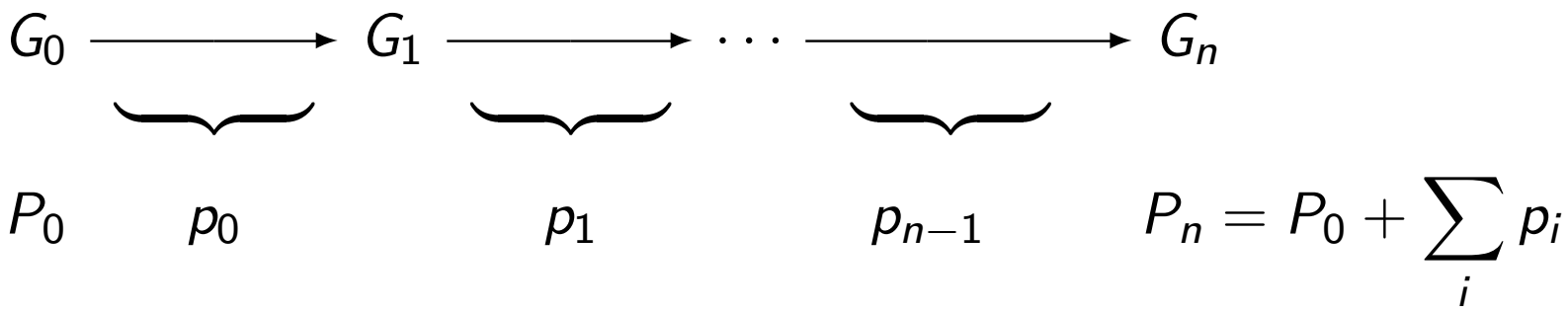

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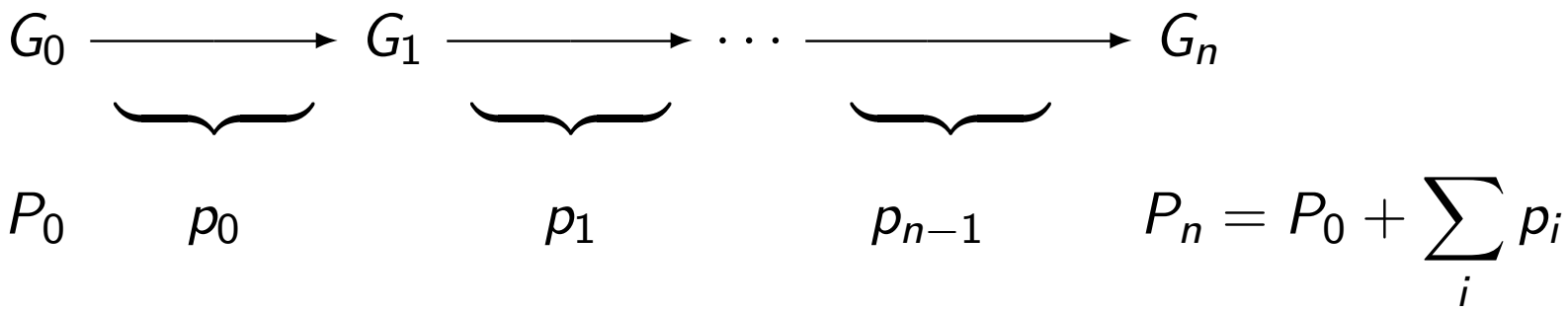

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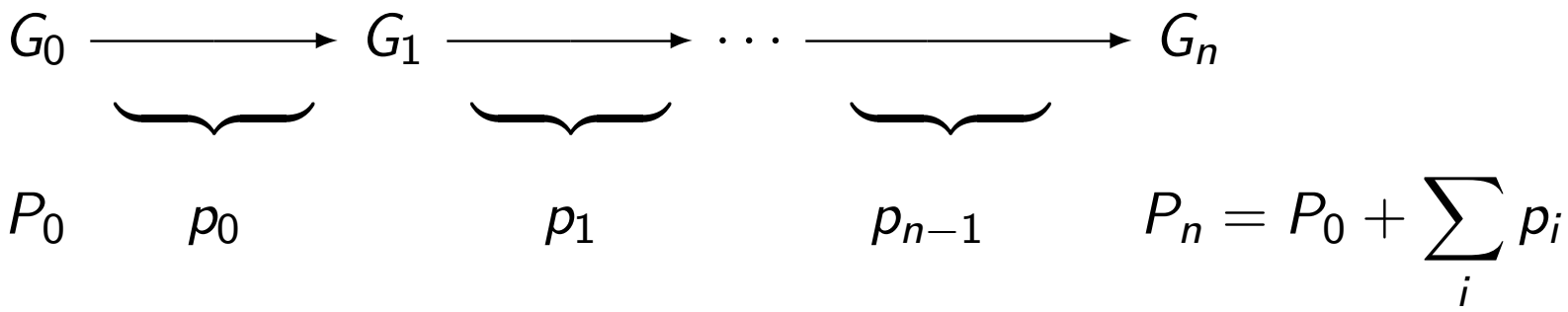

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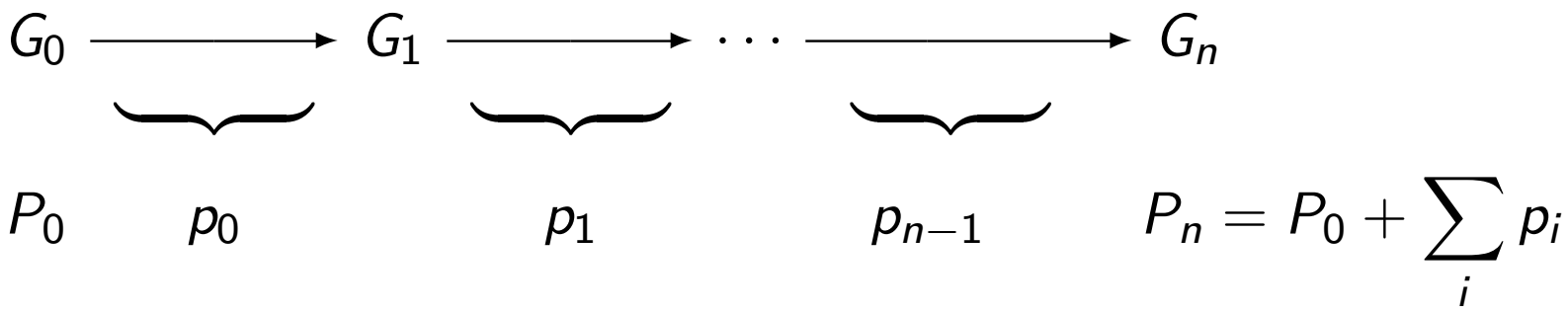

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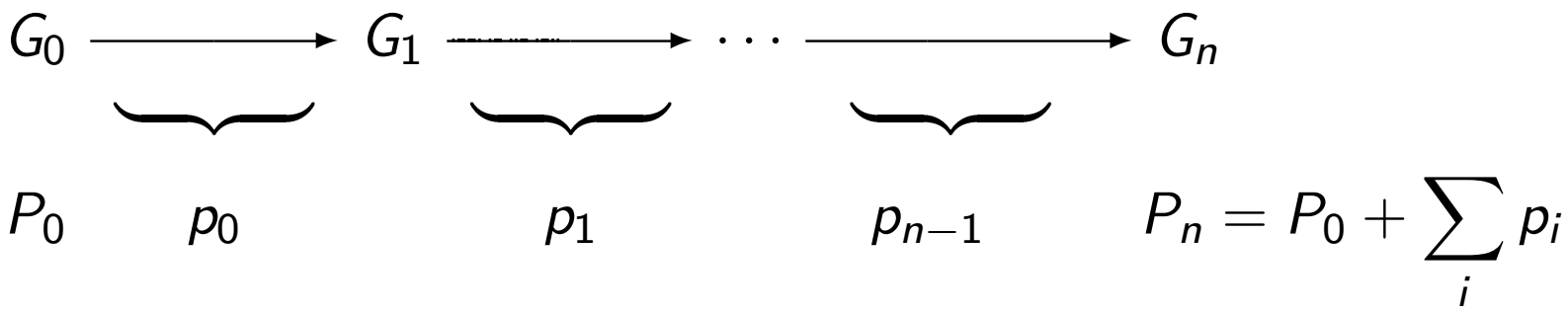

- Sequence of "probabilistic programs"
- Security definitions are probabilistic programs
- Proof steps are program transformations

Towards Automation of Game-playing

CryptoVerif, Blanchet & Pointcheval, 2006

- Games as processes
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Our approach

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- A probabilistic imperative language implemented on Coq
- All transformations justified inside Coq (fully mechanized)

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An Example

PRP/PRF Switching Lemma using Fundamental Lemma

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A Probabilistic Imperative Language I

Probabilistic Execution States

- Deterministic states : lists of (variable, value)
- Oracles : indexed tables of (key, value)
- Probabilistic states : Distributions of (dstate, oracle)
- Events : boolean valued functions on dstates
- Operations on distributions : sum, filter, map, scale, fork
- Probability : `Pr e d = sum (filter e d)`

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$$((p_0, (x, 0)) :: (p_1, (x, 1)) :: nil$$

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A Probabilistic Imperative Language II

Commands

```
Inductive cmd :  Set :=
| skip :  cmd
| assign :  var -> expr -> cmd
| sample_n :  var -> nat -> cmd
| sample_b :  var -> R -> cmd
| find_value :  var -> expr -> cmd
| insert :  expr -> expr -> cmd
| ifte :  expr -> cmd -> cmd -> cmd
| seq :  cmd -> cmd -> cmd
| call :  fun_id -> cmd
```

Example of a Program

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Definition PRF' bad A i :=
 x <- int_e (A i) ;
 y <-$- n ;
 find_value z (var_e y) ;
 ifte (var_e z)
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 insert (var_e x) (var_e y).
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- add entry (x, y) to the oracle

A Probabilistic Imperative Language III

Operational Semantics in terms of probabilistic states

Defined as a relation on

- environments
- probabilistic states (start state)
- commands
- probabilistic states (end state)

written as `prg ||- start -- cmd --> end`

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Boolean sampling `sample_b`

```
nil ||- st0 -- y <-b- p -->
        fork ((p, update y 0)::(1-p, update y 1)::nil) st
```

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$$= | (Pr(E_1 \wedge F_1) + Pr(E_1 \wedge \overline{F}_1)) - (Pr(E_2 \wedge F_2) + Pr(E_2 \wedge \overline{F}_2)) |$$

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Two games are "identical-until-bad" if they are syntactically the same except the branches after `bad <- int_e 1`

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g2

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```

Corollary

Let E_1, E_2 be probabilistic events. Then

$$| Pr_{g1}(E_1) - Pr_{g2}(E_2) | \leq Pr_{g1}(\text{"bad is set 1"})$$

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Lemma

If the adversary makes at most q queries, then

$$| Pr_{\rho}(\mathcal{A} = 1) - Pr_{\sigma}(\mathcal{A} = 1) | \leq \frac{q(q-1)}{2^{n+1}}$$

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Switching Lemma : Game0 (PRP Game)

A game that simulates a random permutation :

```
Definition PRP' bad A i :=
 x <- int_e (A i) ;
 y <-$- n ;
 find_value z (var_e y) ;
 ifte (var_e z)
 (bad <- int_e 1; retry) skip ;
 insert (var_e x) (var_e y).
```

If the randomly selected value is already in the oracle, `bad` is set. Then execute `retry` to select a fresh value.

Game0

```
Definition PRP bad q A := loop q (PRP' bad A)
```

Switching Lemma : Game1 (PRF Game)

A game that simulates a random function :

```
Definition PRF' bad A i :=
 x <- int_e (A i) ;
 y <-$- n ;
 find_value z (var_e y) ;
 ifte (var_e z)
 (bad <- int_e 1; skip) skip ;
 insert (var_e x) (var_e y).
```

No action after setting bad.
Behaves as a function, not permutation (Collisions are allowed).

Game1

```
Definition PRF bad q A := loop q (PRF' bad A)
```

Outline

Sketch of the Proof : Step 1

Switching Lemma

If the adversary makes at most q queries, then

$$| Pr_{Game0}(\mathcal{A} = 1) - Pr_{Game1}(\mathcal{A} = 1) | \leq \frac{q(q-1)}{2^{n+1}}$$

Sketch of the Proof : Step 1

Step 1 : Application of a version of Fundamental Lemma to Game0 and Game1

Corollary

$$| Pr_{Game0}(\mathcal{A} = 1) - Pr_{Game1}(\mathcal{A} = 1) | \leq Pr_{Game0}(\text{“bad is set 1”})$$

Sketch of the Proof : Step 2

Step 2 : Evaluate the bound $Pr_{Game0}(\text{“bad is set 1”})$ by induction on q

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Step 2 : Evaluate the bound $Pr_{Game0}(\text{“bad is set 1”})$ by induction on q

Lemma

$$Pr_{Game0}(\text{“bad is set 1”}) \leq \frac{q(q-1)}{2^{n+1}}$$

Switching Lemma

If the adversary makes at most q queries, then

$$| Pr_{Game0}(\mathcal{A} = 1) - Pr_{Game1}(\mathcal{A} = 1) | \leq \frac{q(q-1)}{2^{n+1}}$$

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Reference

R. Affeldt, M. Tanaka, N. Marti.
Formal Proof of Provable Security
by Game-playing in a Proof Assistant.
ProvSec 2007, LNCS 4784, pages 151–168.