Probabilistic Automata: Probability and Nondeterminism Together

Roberto Segala
University of Verona

Objectives

- Why/how probability and nondeterminism?
- What are the key ingredients?
  - Measures and measurable functions
  - Conservative extensions of Labeled Transition Systems
- Probabilistic Automata
  - Definition
  - Comparison with other models
  - Parallel composition
- Bisimulation relations
  - Strong, strong probabilistic, weak
- Language inclusion
- Highlights of a Case study

Why Nondeterminism with Probability? Distributed Algorithms

- Some problems are unsolvable
  - Consensus [FLP85]
- … but are solvable with randomization
  - Probabilistic consensus [Ben83,AH90]
- Probability and nondeterminism coexist
  - Probability:
    - Processes flip coins
  - Nondeterminism:
    - Several processes in parallel
    - Do not care whether the coin is fair
- Quantitative analysis
  - What is the worst expected complexity?

Why Nondeterminism with Probability? Stochastic Games

- Nondeterminism
  - Each player has several moves available
- Probability
  - Moves may involve coin flipping
- Quantitative analysis
  - What is the best probability to win the game?

Why Nondeterminism with Probability? Security

- Nondeterminism
  - User behavior (adversary in Dolev-Yao)
  - Relative speeds of agents
  - Agent behavior (usually deterministic)
- Probability
  - Users and agents flip coins
    - Nonces, keys, random protocols
- Quantitative analysis
  - Probability of attack (negligible)

Why Nondeterminism with Probabililty? Concurrency Theory

- Nondeterminism
  - Scheduling within parallel composition
  - Unknown behavior of the environment
  - Underspecification

- Probability
  - Environment may be stochastic
  - Processes may flip coins

Probability and Nondeterminism: How?

- Reactive, Generative Systems [LS89,GSST90]
  - Labeled transition systems
    - Add probabilities to the arcs
  - Process algebras
    - Replace + with probabilistic +
- Probabilistic Automata [Seg95]
  - Labeled transition systems
    - Replace target states with target measures in transitions
  - Process Algebras
    - Add a probabilistic + operator (named $\oplus$)

Automata

$$A = (Q\ ,\ q_0\ ,\ E\ ,\ H\ ,\ D)$$

- Transition relation $D \subseteq Q \times (E \cup H) \times Q$
- Internal (hidden) actions
- External actions: $E \cap H = \emptyset$
- Initial state: $q_0 \in Q$
- States

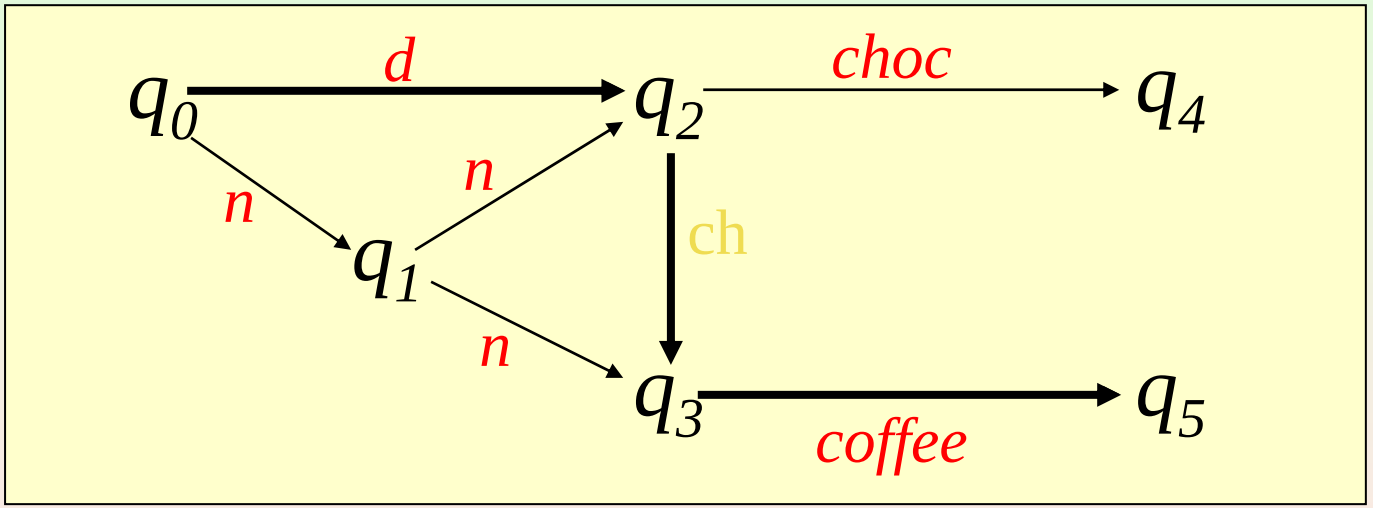
Probabilistic Automata

$$PA = (Q\ ,\ q_0\ ,\ E\ ,\ H\ ,\ D)$$

- Transition relation $D \subseteq Q \times (E \cup H) \times \mathrm{Disc}(Q)$
- Internal (hidden) actions
- External actions: $E \cap H = \emptyset$
- Initial state: $q_0 \in Q$
- States

Example: Automata

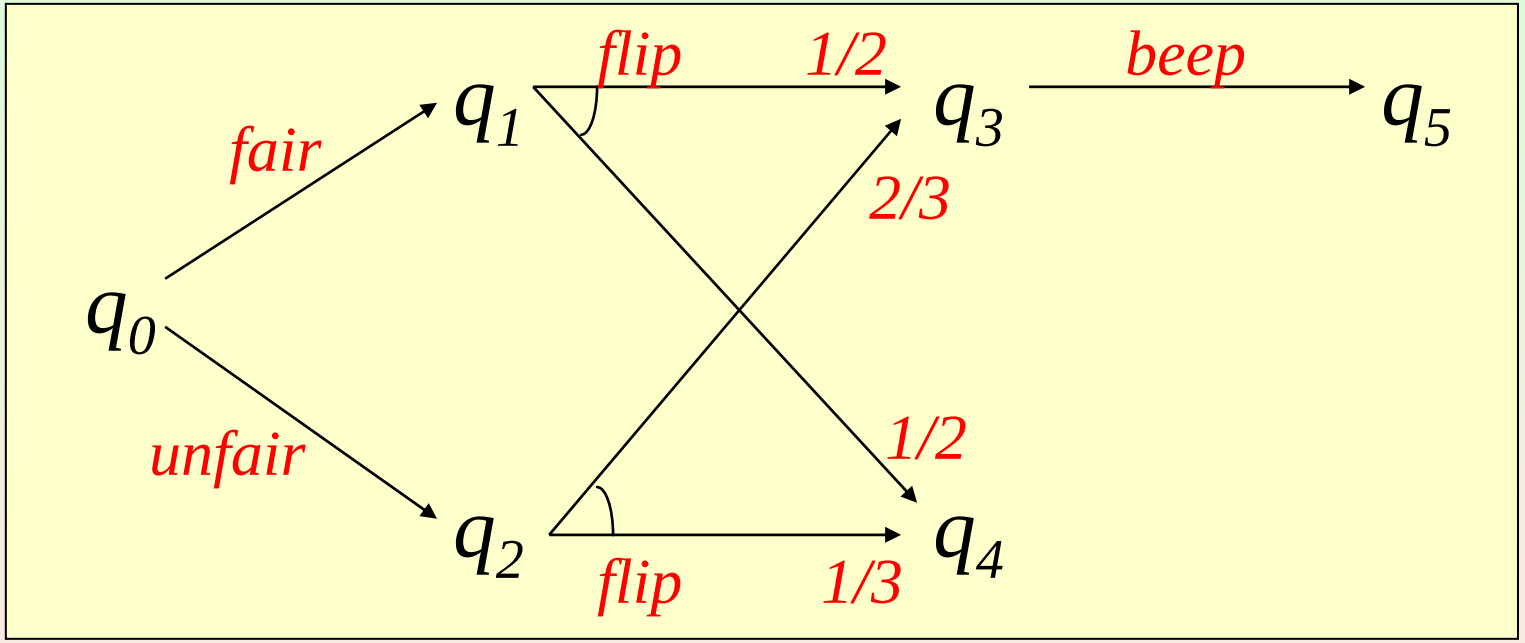
$$A = (Q\ ,\ q_0\ ,\ E\ ,\ H\ ,\ D)$$



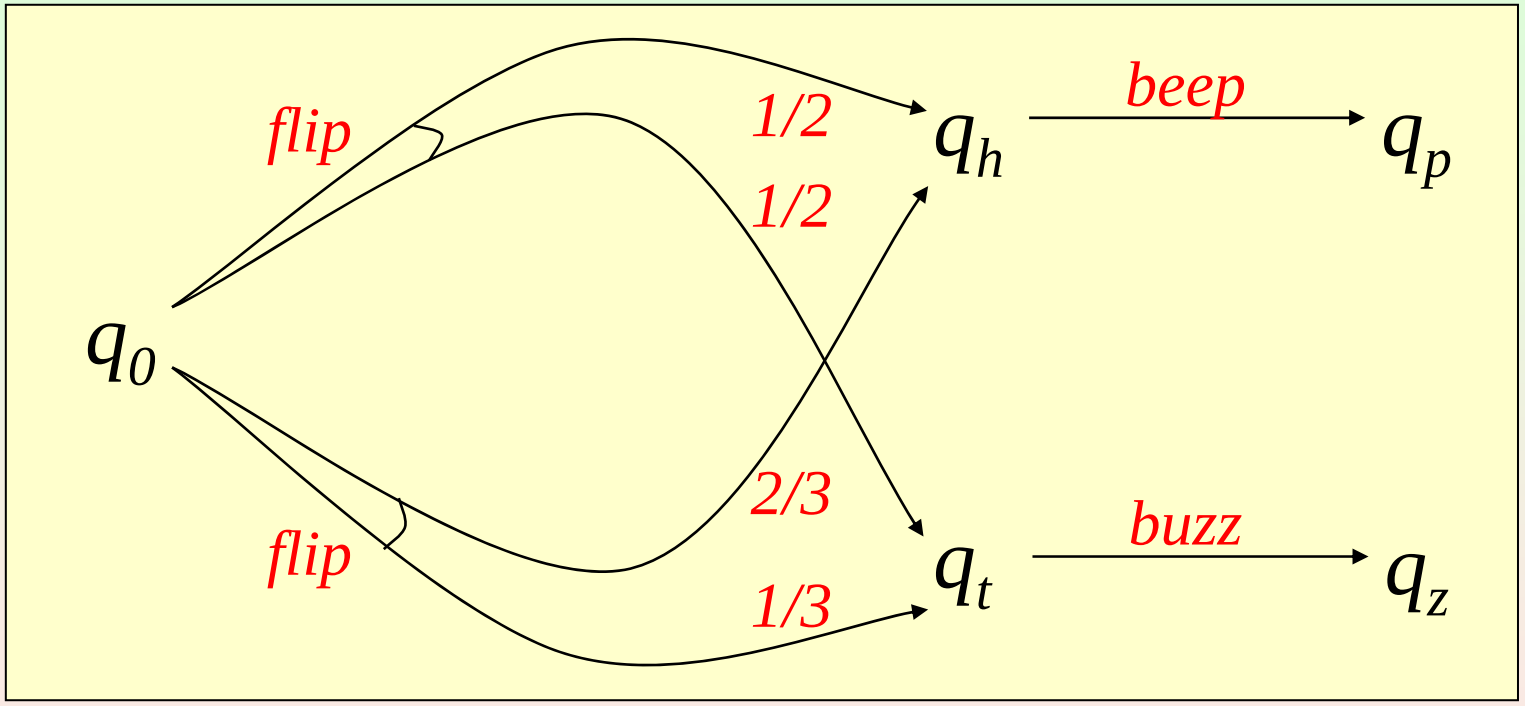
Execution: $q_0\ n\ q_1\ n\ q_2\ ch\ q_3\ coffee\ q_5$

Trace: *n n coffee*

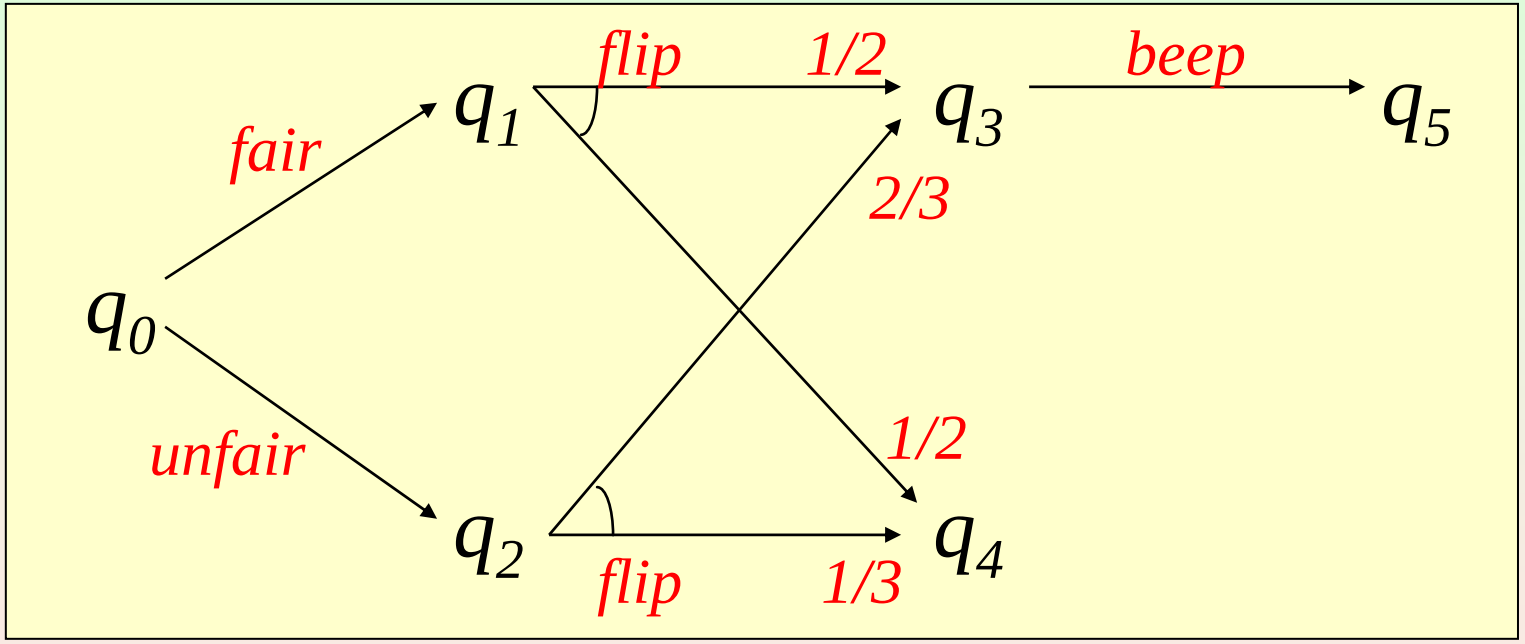
Example: Probabilistic Automata



Example: Probabilistic Automata

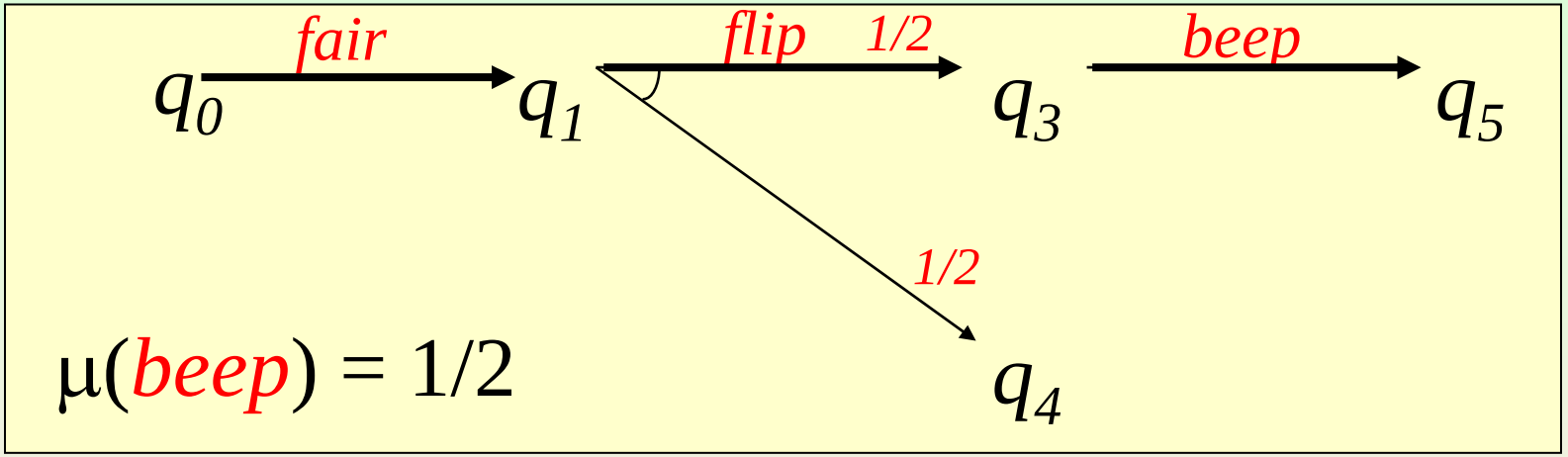


Example: Probabilistic Automata



What is the probability of beeping?

Example: Probabilistic Executions



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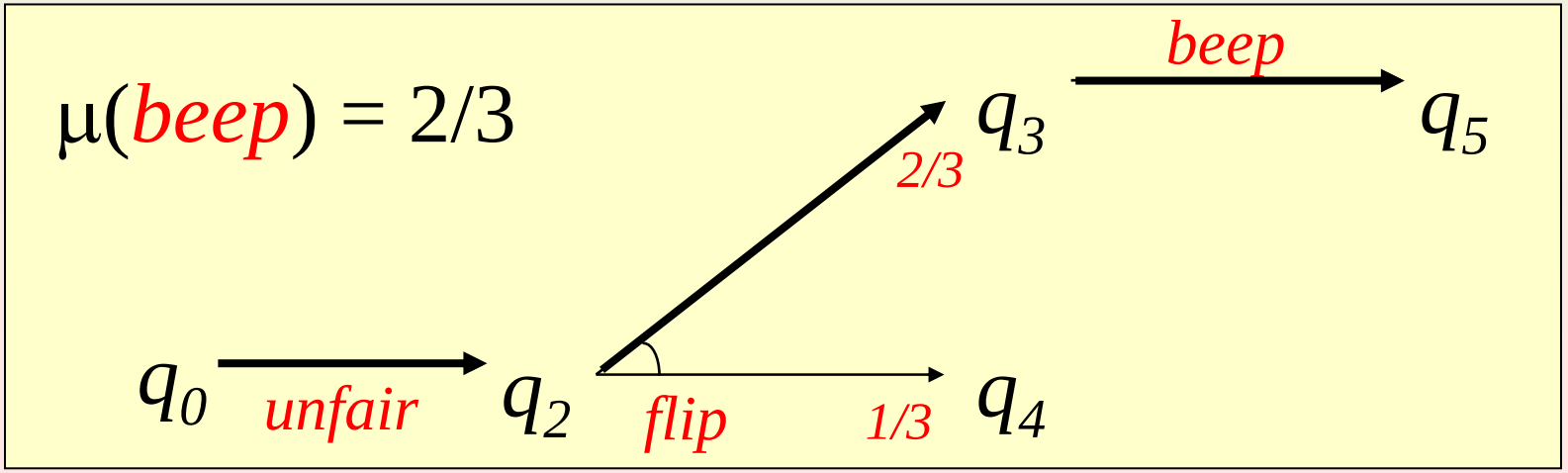

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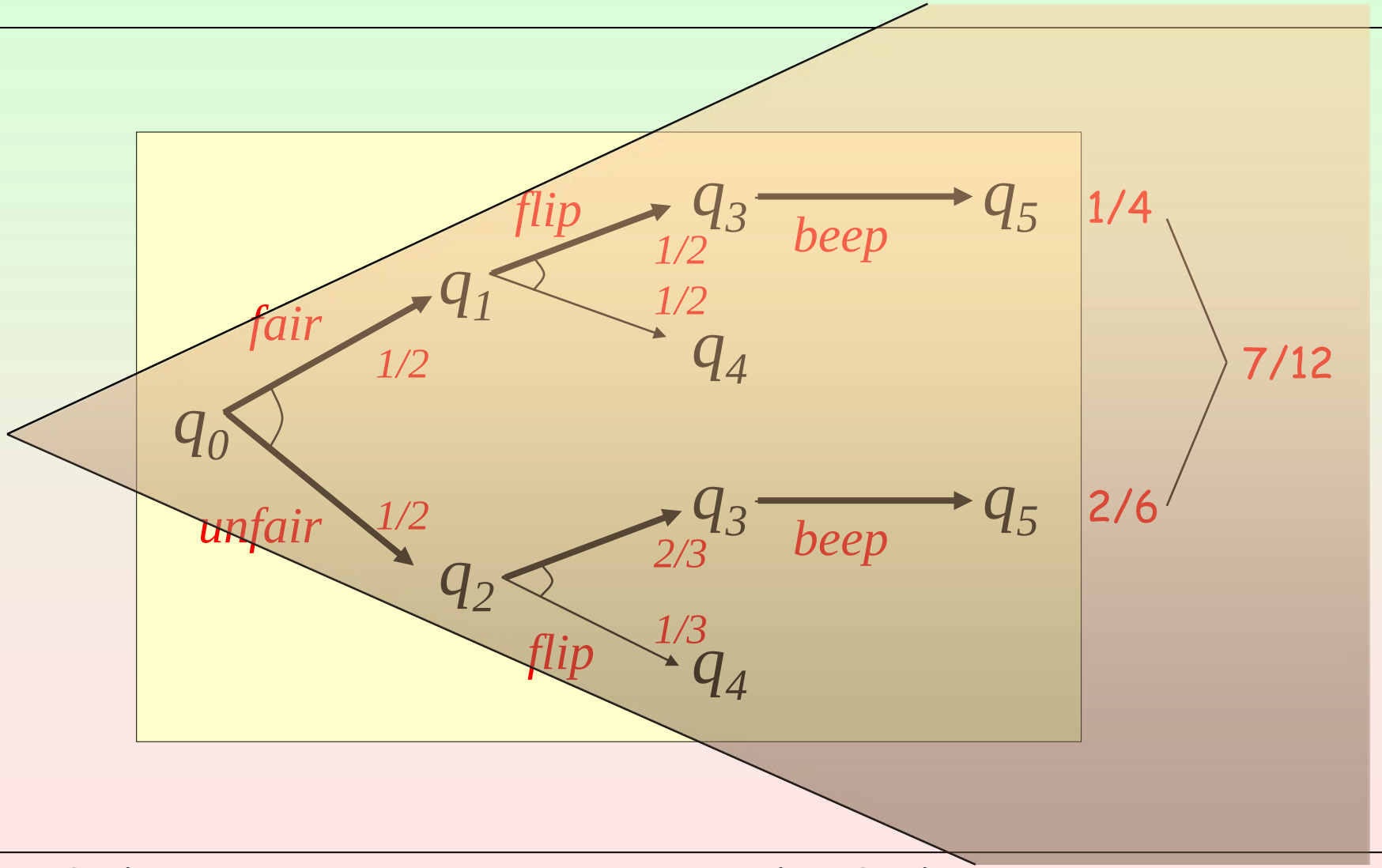
Example: Probabilistic Executions



Measure Theory

- Sample set
  - Set of objects Ω
- Sigma-field (σ-field)
  - Subset F of 2^{Ω} satisfying
    - Inclusion of Ω
    - Closure under complement
    - Closure under countable union
    - Closure under countable intersection
- Measure on (Ω,F)
  - Function μ from F to $\Re^{\geq 0}$
    - For each countable collection $\{X_i\}_I$ of pairwise disjoint sets of F, $\mu(\cup_I X_i) = \Sigma_I \mu(X_i)$
- (Sub-)probability measure
  - Measure μ such that $\mu(\Omega) = 1$ ($\mu(\Omega) \leq 1$)
- Sigma-field generated by $C \subseteq 2^{\Omega}$
  - Smallest σ-field that includes C

Why not $F = 2^{\Omega}$?

Example: set of executions

Flip a fair coin infinitely many times

$\Omega = \{h,t\}^{\infty}$

Study probabilities of sets of executions

$\mu(\omega) \neq 0$ for each $\omega \in \Omega$

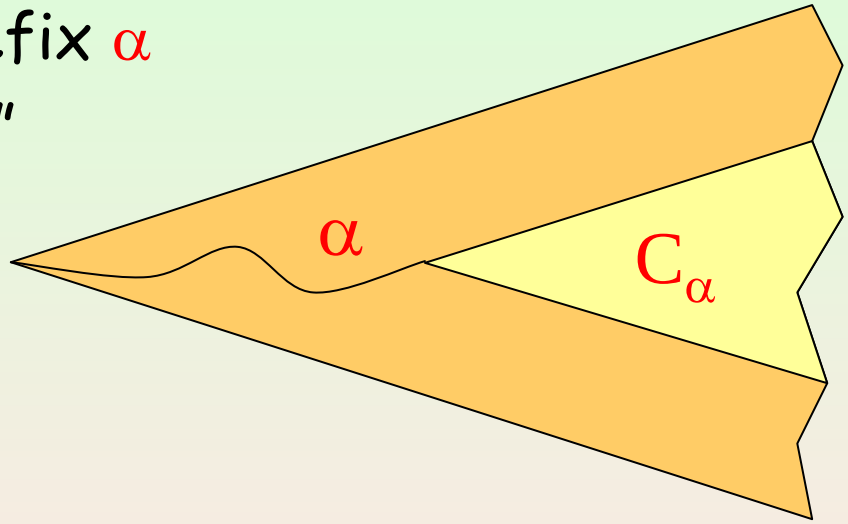
$\mu(\text{first coin } h) = 1/2$

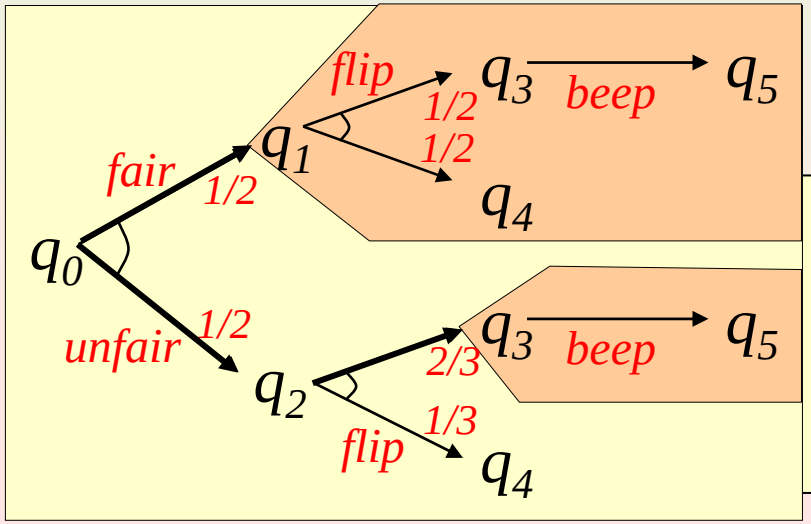
which sets can I measure?

Theorem: there is no probability measure on 2^{Ω} such that $\mu(\omega) = 0$ for each $\omega \in \Omega$.

Cones and Measures

- Cone of α
  - Set of executions with prefix α
  - Represent event "α occurs"
- Measure of a cone
  - Product edges of α





extends uniquely

σ-field generated by cones

Examples of Events

- Eventually action a occurs
  - Union of cones where action a occurs once
- Action a occurs at least n times
  - Union of cones where action a occurs n times
- Action a occurs at most n times
  - Complement of action a occurs at least n+1 times
- Action a occurs exactly n times
  - Intersection of previous two events
- Action a occurs infinitely many times
  - Intersection of action a occurs at least n times for all n
- Execution α occurs and nothing is scheduled after
  - Set consisting of α only
  - C_α intersected complement of cones that extend α

Schedulers - Probabilistic Executions

Scheduler

Function $\sigma : exec^*(A) \rightarrow SubDisc(D)$

if $\sigma(\alpha)((q,a,\nu)) > 0$ then $q = lstate(\alpha)$

Probabilistic execution generated by σ from state r

Measure $\mu_{\sigma,r}$

$$\mu_{\sigma,r}(C_s) = 0 \quad \text{if} \quad r \neq s$$

$$\mu_{\sigma,r}(C_r) = 1$$

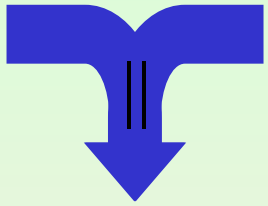
$$\mu_{\sigma,r}(C_{\alpha a q}) = \mu_{\sigma,r}(C_\alpha) \cdot \left(\sum_{(s,a,\nu) \in D} \sigma(\alpha)((s,a,\nu))\nu(q) \right)$$

Other Models

- Reactive and generative systems
  - Restricted forms of transitions
- Labeled Concurrent Markov Chains
  - Restricted forms of transitions
- Rabin's Probabilistic Automata
  - Introduced in the context of language theory
  - Extended by our Probabilistic Automata
- Unlabeled systems [Var85,BA95,BK98]
  - Can be Probabilistic Automata with a single invisible action
  - Labels may be associated with states
  - The theory does not change
- Markov Chains
  - Unlabeled systems that enable one transition from each state
- Probabilistic Input/Output Automata
  - Add Input/Output distinction on actions
  - Useful to handle composition of generative PAs

Composition of Probabilistic Automata

$$A_1 = (Q_1, q_1, E_1, H_1, D_1) \qquad A_2 = (Q_2, q_2, E_2, H_2, D_2)$$

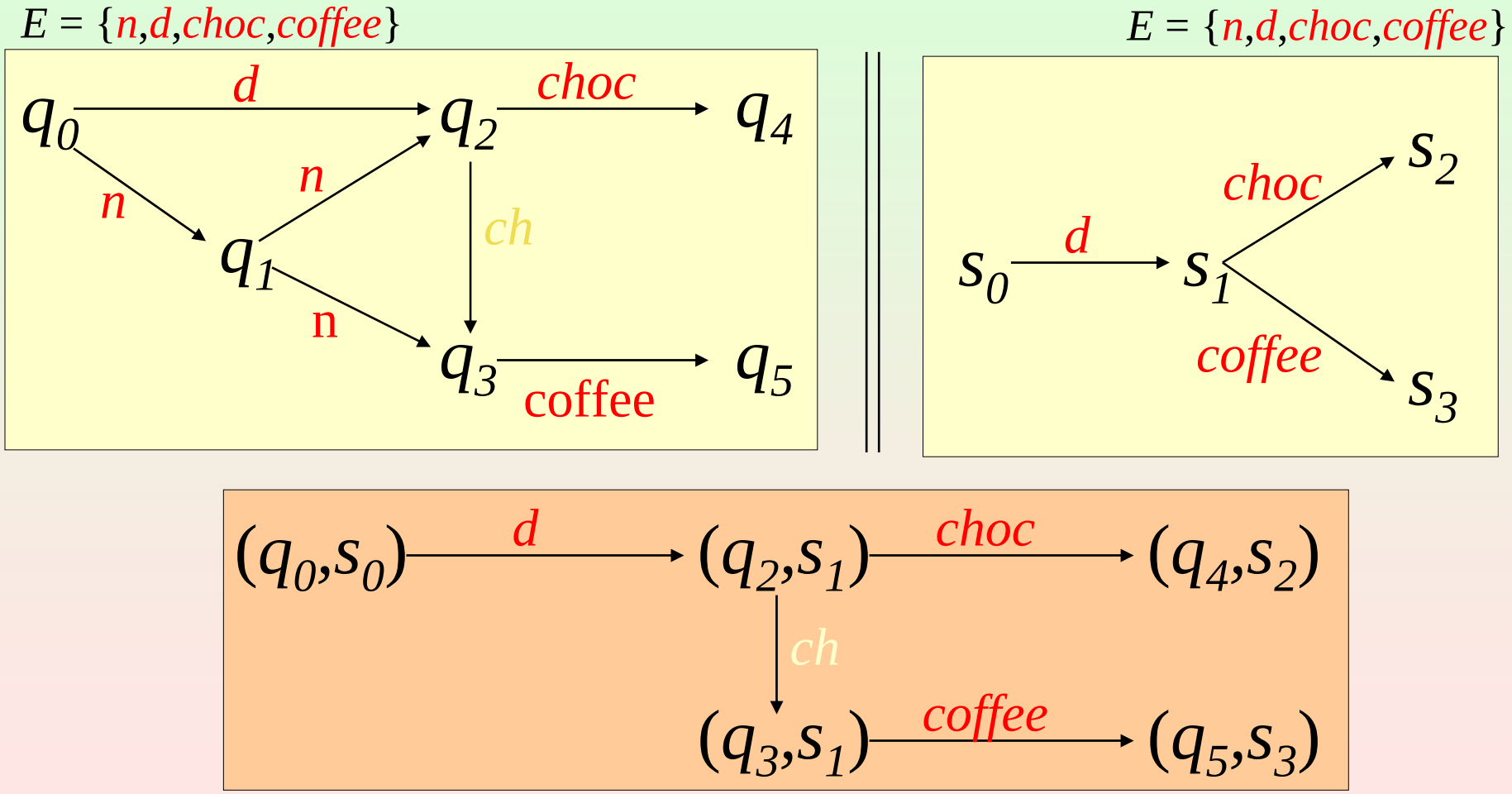


$$A_1 \| A_2 = (Q_1 \acute{} Q_2 , (q_1,q_2) , E_1 \cup E_2 , H_1 \cup H_2 , D)$$

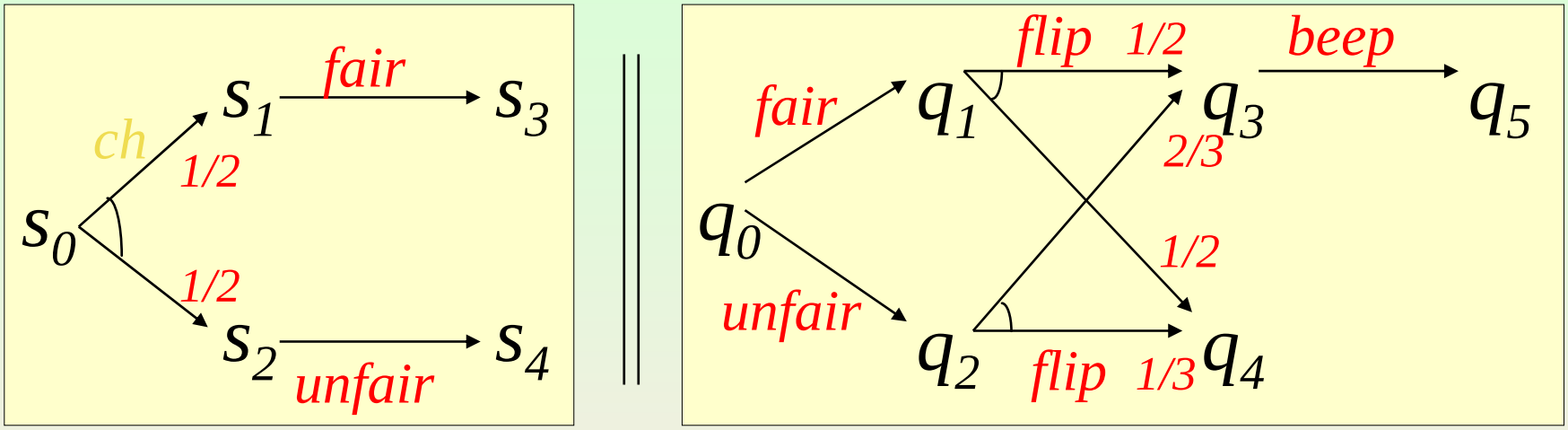
$$D = \left\{ (q,a,(s_1,s_2)) \;\middle|\; \begin{array}{l} \text{if } a \in E_i \cup H_i \text{ then } (\pi_i(q) , a , s_i) \in D_i \\ \text{if } a \notin E_i \cup H_i \text{ then } s_i = \pi_i(q) \end{array} \quad i \in \{1,2\} \right\}$$

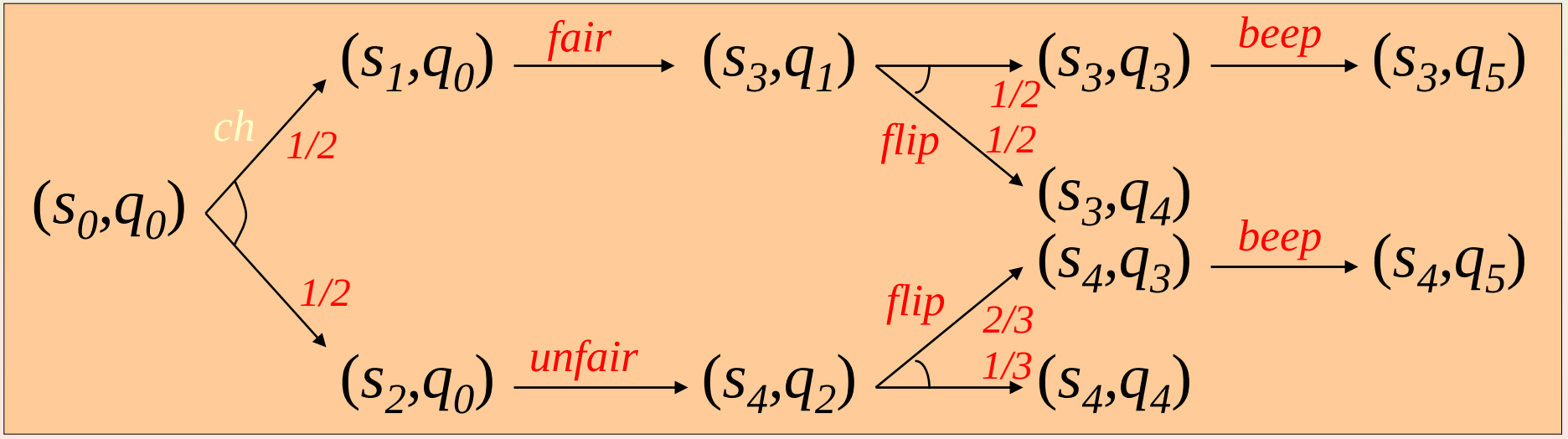
$$D = \left\{ (q,a,\mu_1 \times \mu_2) \;\middle|\; \begin{array}{l} \text{if } a \in E_i \cup H_i \text{ then } (\pi_i(q) , a , \mu_i) \in D_i \\ \text{if } a \notin E_i \cup H_i \text{ then } \mu_i = \delta(\pi_i(q)) \end{array} \quad i \in \{1,2\} \right\}$$

Example: Composition of Automata



Ex. Composition of Probabilistic Automata





Projections

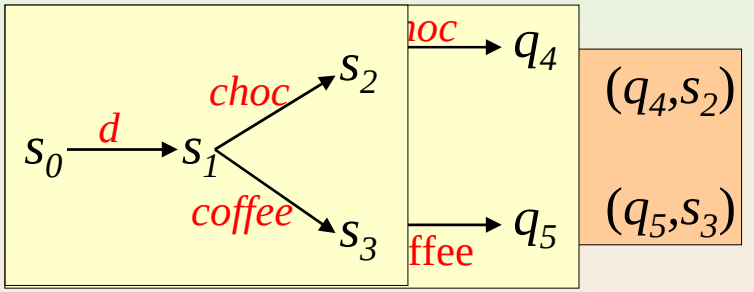
Let α be an execution of $A_1 \| A_2$

$\alpha = (q_0,s_0)\ d\ (q_2,s_1)\ ch\ (q_3,s_1)\ coffee\ (q_5,s_3)$

What are the contributions of A_1 and A_2?

$\pi_1(\alpha) \equiv q_0\ d\ q_2\ ch\ q_3\ coffee\ q_5$

$\pi_2(\alpha) \equiv s_0\ d\ s_1\ coffee\ s_3$

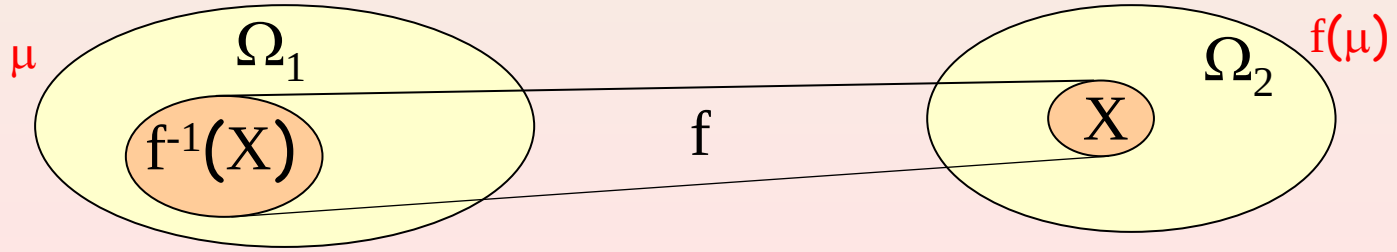


Theorem

$\alpha \in execs(A_1||A_2)$ iff $\forall_{i \in \{1,2\}}\ \pi_i(\alpha) \in execs(A_i)$

Measure Theory: Image Measure

- Measurable function from (Ω_1,F_1) to (Ω_2,F_2)
  - Function f from Ω_1 to Ω_2
  - For each element X of F_2, $f^{-1}(X) \in F_1$
- Image measure $f(\mu)$
  - $f(\mu)(X) = \mu(f^{-1}(X))$



Projections

The projection function is measurable

$\pi(\mu)$: image measure under π of μ

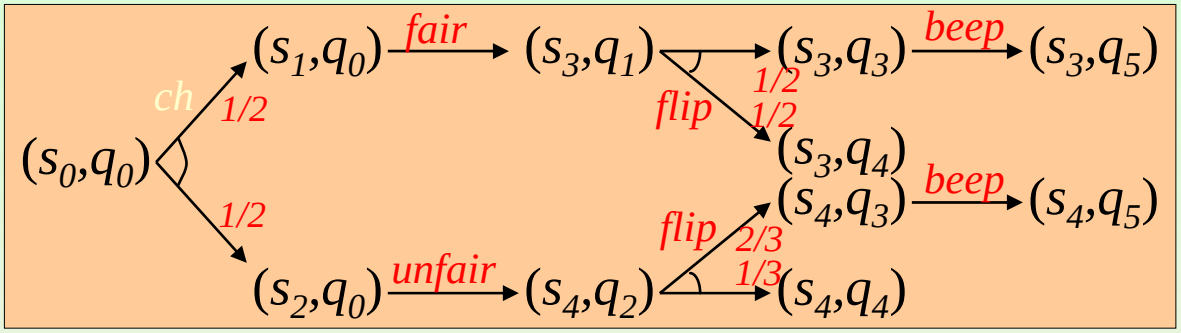
Theorem

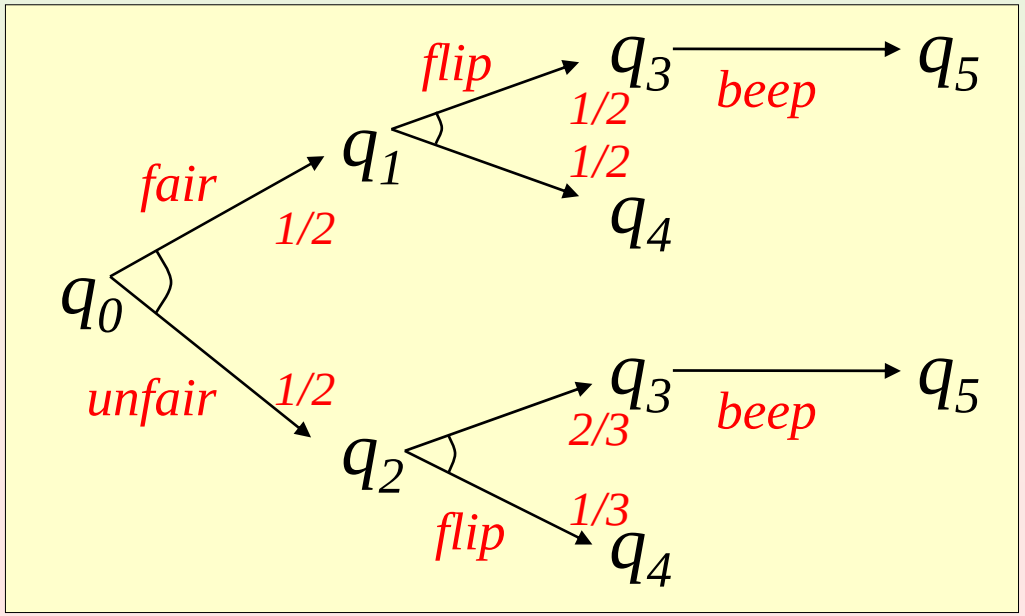
If μ is a probabilistic execution of $A_1 \| A_2$ then

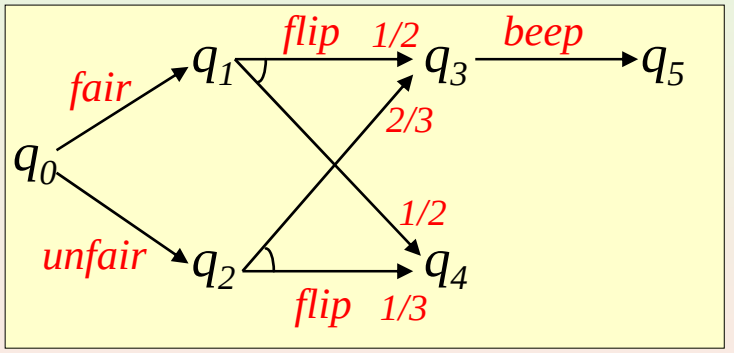
$\pi_i(\mu)$ is a probabilistic execution of A_i

Example: Projection

Projection onto right component







Note that the scheduler is randomized

Use of Projections

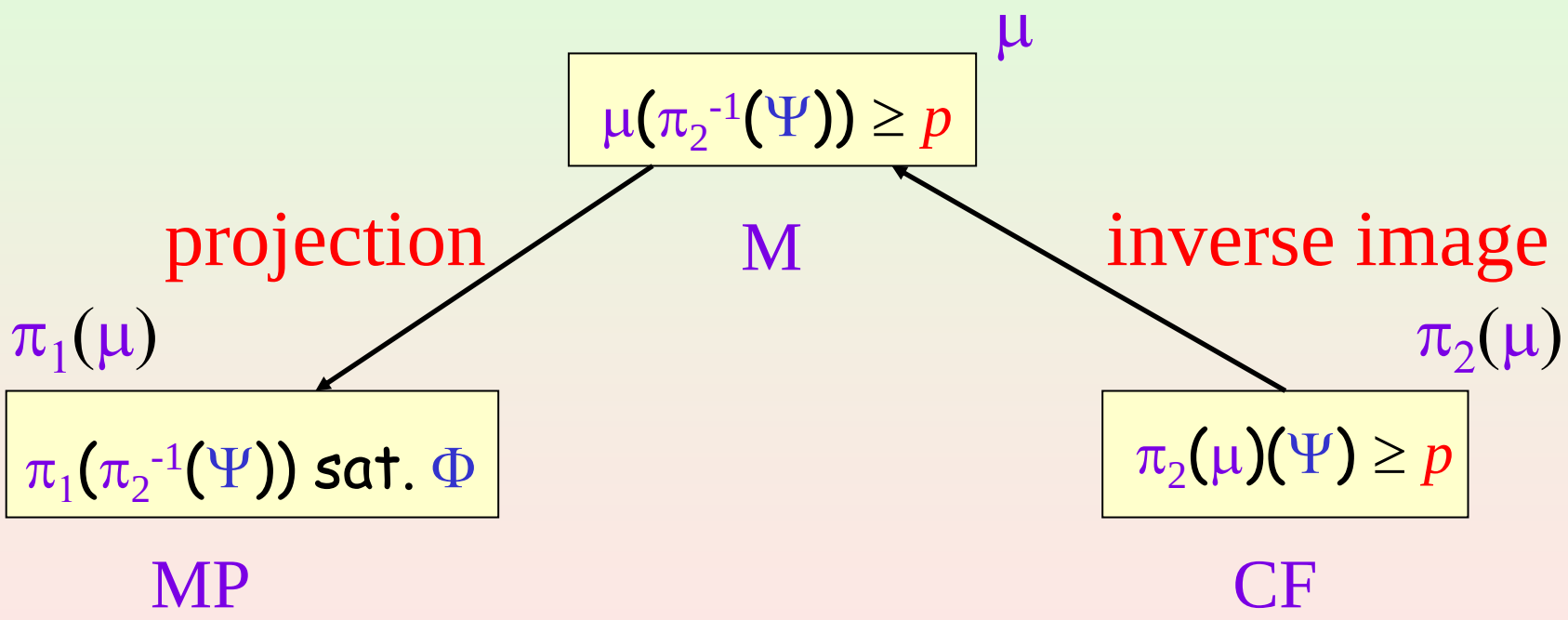
- Let M = MP||CF
- Suppose that MP satisfies Φ provided that the environment (CF) satisfies Ψ
- Suppose that CF satisfies Ψ with probability p
- Can I conclude that M satisfies Φ with probability p?

$$\frac{MP \models \Psi \Rightarrow \Phi \qquad CF \models [\Psi]_{\geq p}}{M \models [\Phi]_{\geq p}}$$

- This example is taken from a real case study [PLS01]
  - Randomized consensus protocol of Aspnes and Herlihy [AH90]
  - MP is a complex non randomized protocol
  - CF is a relatively simple randomized coin flipper

Formal Argument

Let μ be a probabilistic execution of M.



Bisimulation Relations

We have the following objectives

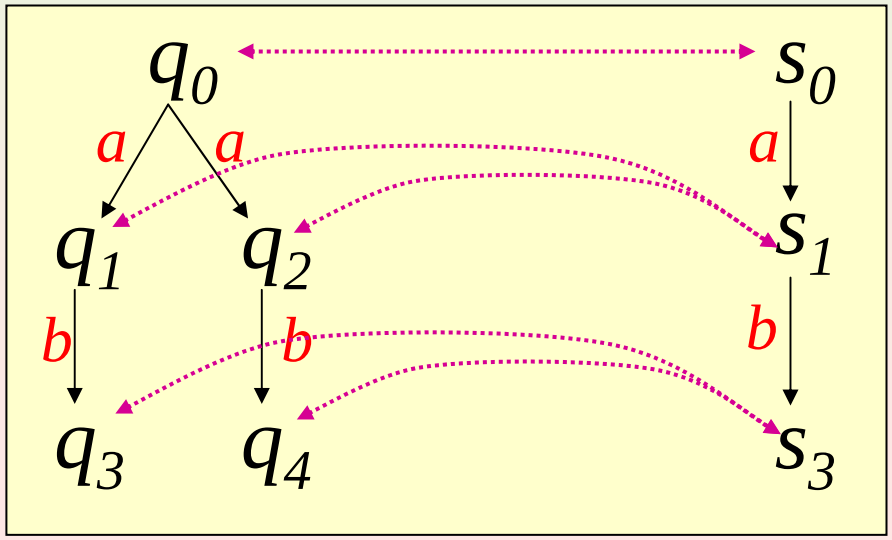
- Same definitional style as for automata
  - Where are the key differences?
  - Keep definitions simple
- Uniform treatment
  - The literature is not uniform
  - This causes a lot of confusion
  - How can we see everything from a single point of view?

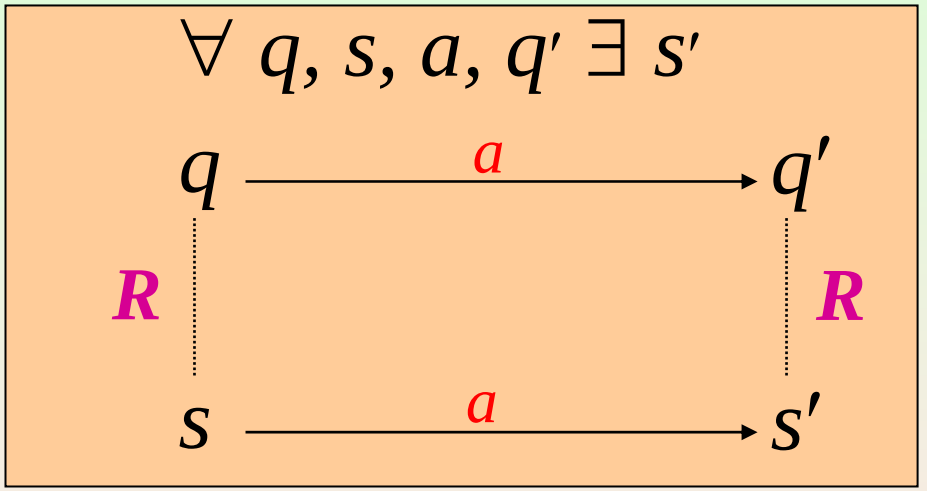
Strong Bisimulation on Automata

Strong bisimulation between A_1 and A_2

Relation $\boldsymbol{R} \subseteq Q \times Q$,

$Q = Q_1 \uplus Q_2$, such that





Strong Bisimulation on Probabilistic Automata

Strong bisimulation between A_1 and A_2

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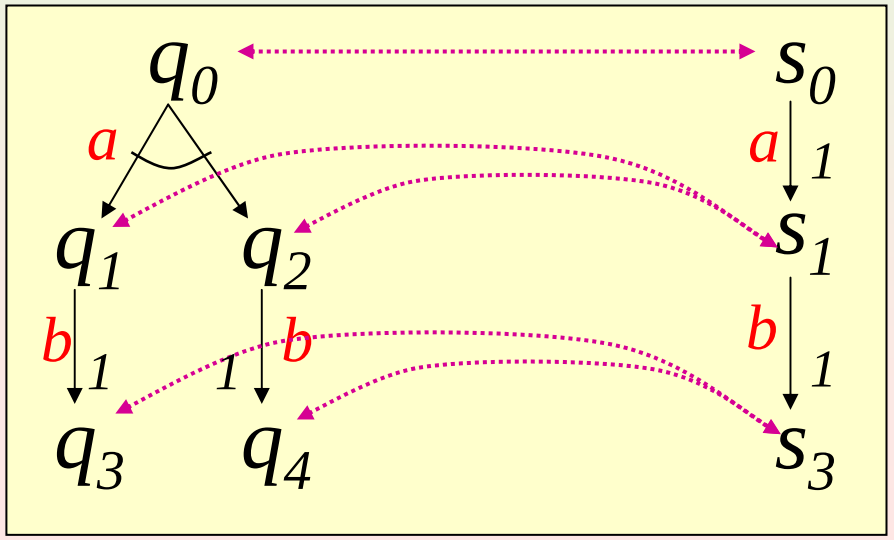


$\forall\, q, s, a, \mu \;\exists\, \mu'$

$q \xrightarrow{a} \mu$

$\boldsymbol{R}$ $\qquad$ $\boldsymbol{R}$

$s \xrightarrow{a} \mu'$

$\mu \;\boldsymbol{R}\; \mu'$ [LS89]

$\Leftrightarrow$

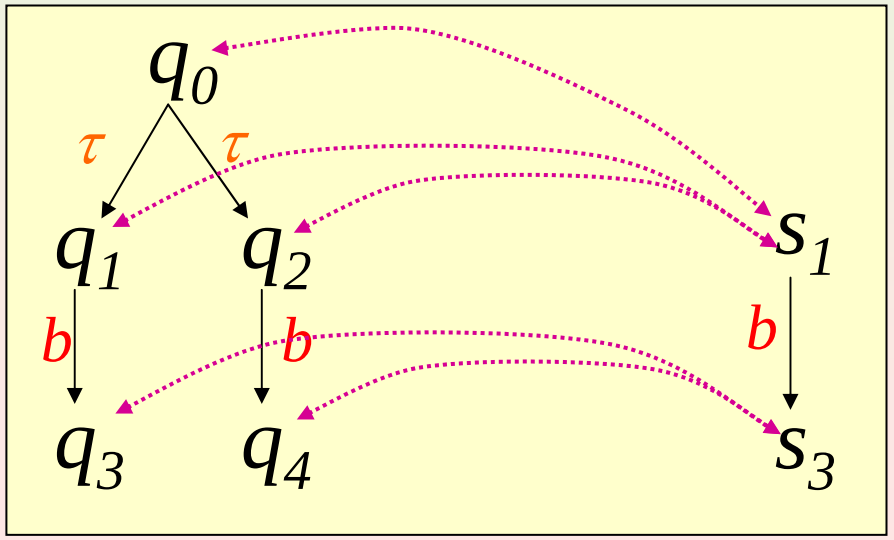
$\forall C \in Q/\boldsymbol{R} \,.\, \mu(C) = \mu'(C)$

Weak Bisimulation on Automata

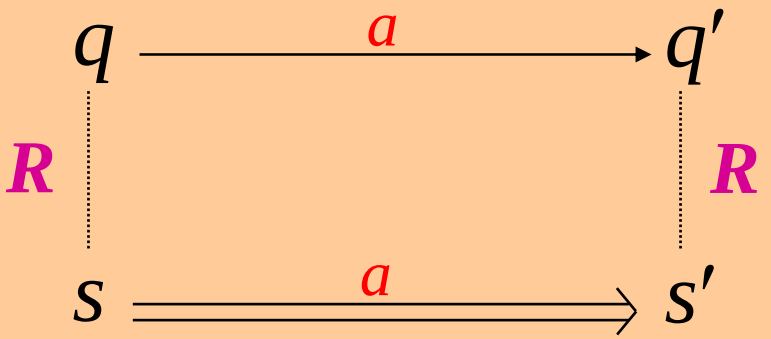
Weak bisimulation between A_1 and A_2

Relation $\boldsymbol{R} \subseteq Q \times Q$,

$Q = Q_1 \uplus Q_2$, such that



$\forall\ q, s, a, q'\ \exists\ s'$



$s \overset{a}{\Rightarrow} s'$

$\Leftrightarrow$

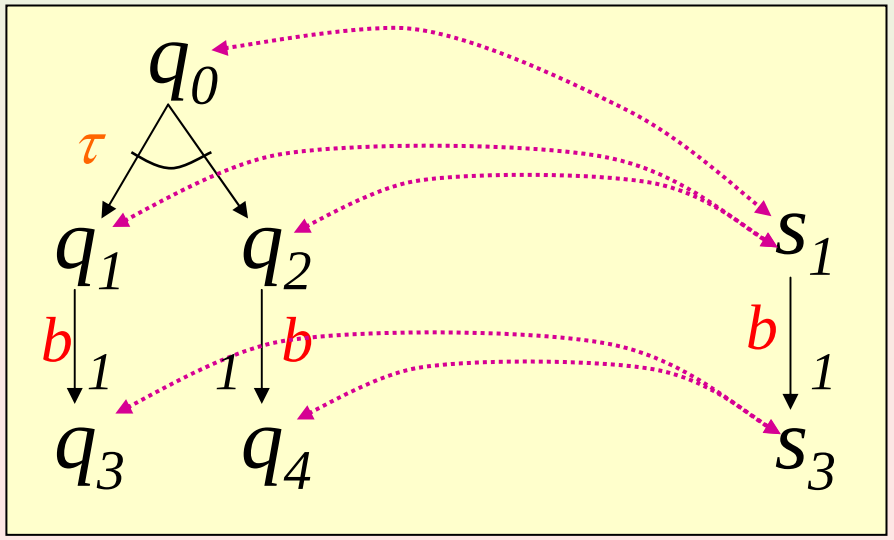
$\exists \alpha$: $trace(\alpha)=a$, $fstate(\alpha)=s$, $lstate(\alpha)=s'$

Weak bisimulation on Probabilistic Automata

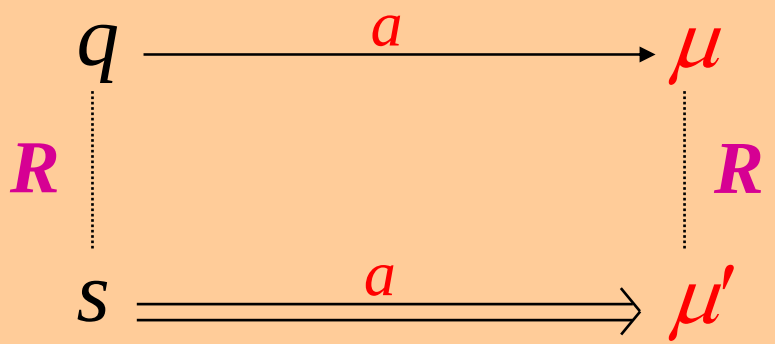
Weak bisimulation between A_1 and A_2

Relation $R \subseteq Q \times Q$,

$Q = Q_1 \uplus Q_2$, such that



$\forall\, q, s, a, \mu \;\exists\, \mu'$



$\mu \; R \; \mu'$ [LS89]

$\Leftrightarrow$

$\forall C \in Q/R \,.\, \mu(C) = \mu'(C)$

Weak Transition

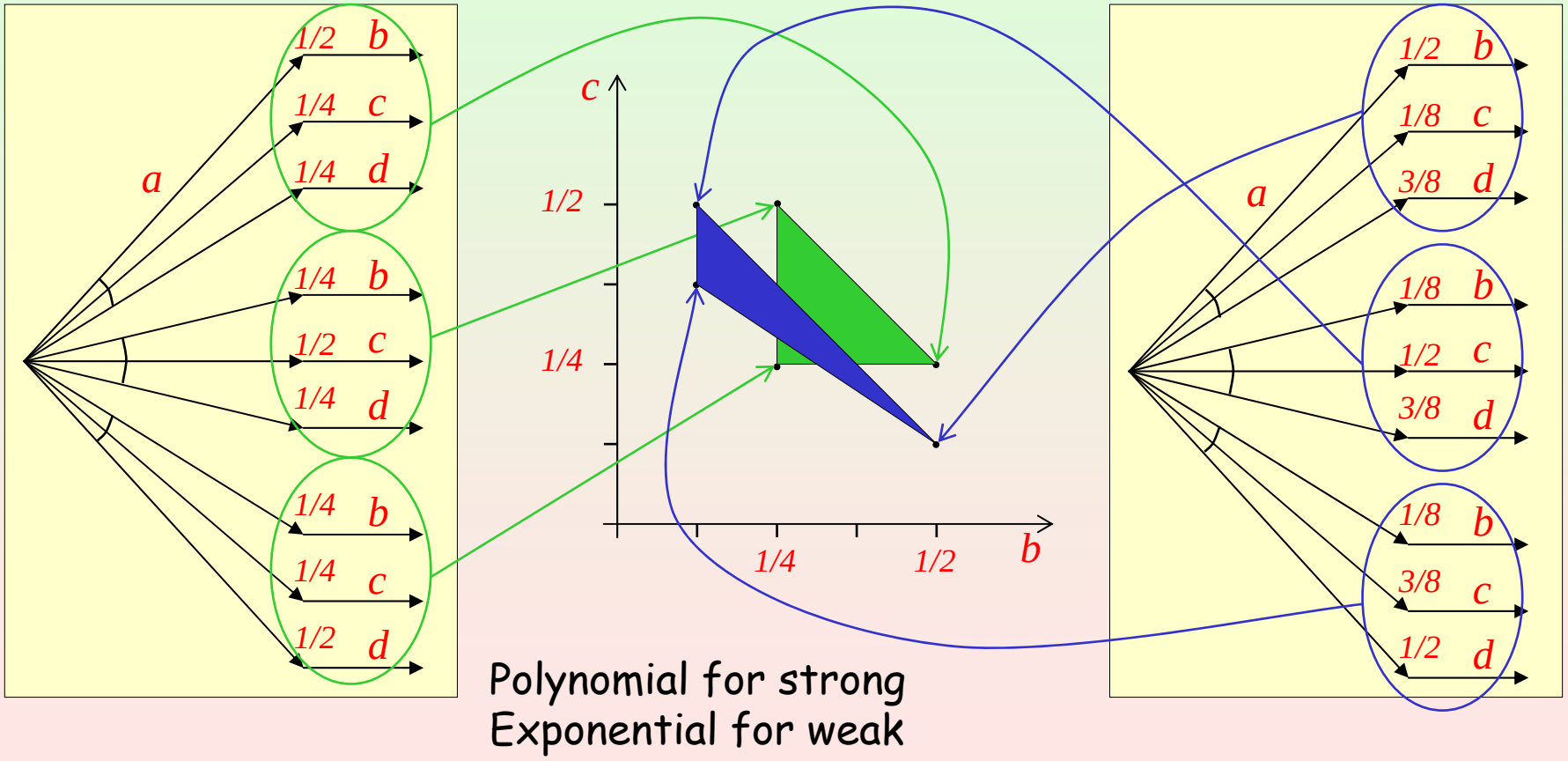
$$q \overset{a}{\Longrightarrow} \rho$$

There is a probabilistic execution μ such that

- $\mu(exec^*) = 1$ (it is finite)
- $trace(\mu) = \delta(a)$ (its trace is a)
- $fstate(\mu) = \delta(q)$ (it starts from q)
- $lstate(\mu) = \rho$ (it leads to ρ)

$q \overset{a}{\Rightarrow} s$ iff $\exists \alpha$: $trace(\alpha)=a$, $fstate(\alpha)=q$, $lstate(\alpha)=s$

Decision Procedures



Polynomial for strong
Exponential for weak

Trace Distributions

The *trace* function is measurable

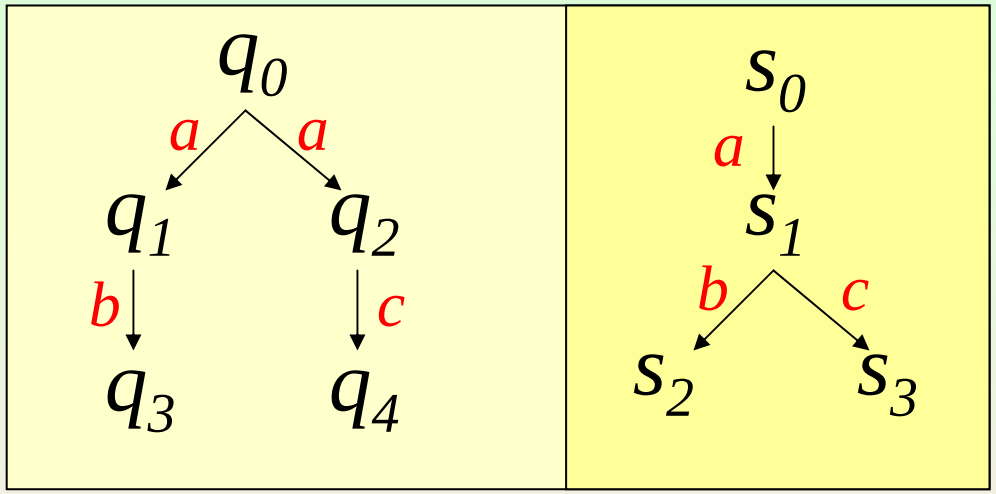
Trace distribution of μ

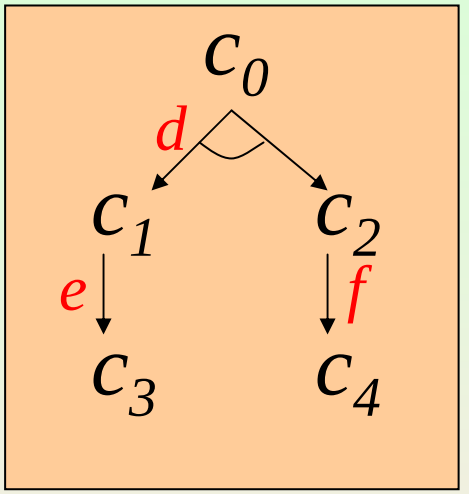
$tdist(\mu)$: image measure under *trace* of μ

Trace distribution inclusion preorder

$A_1 \leq_{TD} A_2$ iff $tdists(A_1) \subseteq tdists(A_2)$

Trace Distribution Inclusion is not Compositional





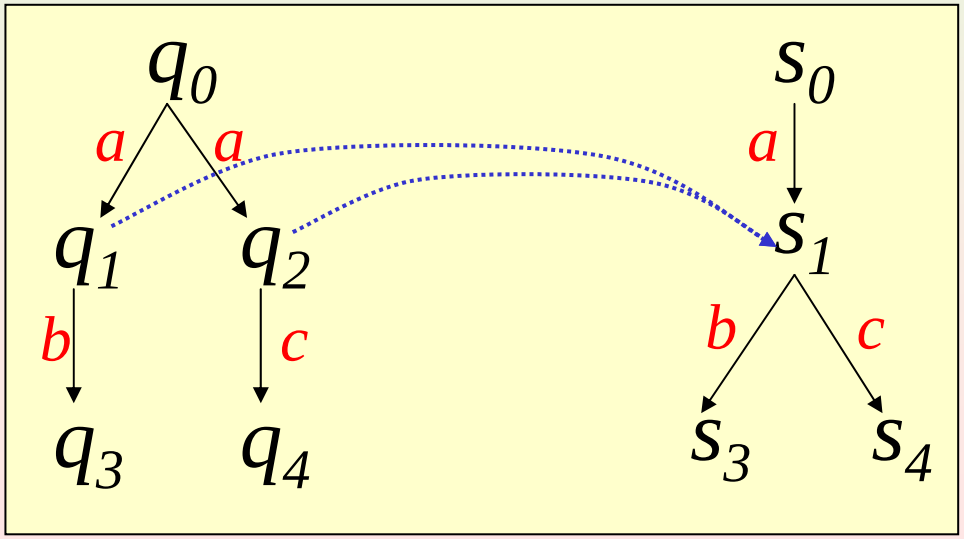
$(s_0,c_0) \xrightarrow{a} (s_1,c_0)$ with d to $(s_1,c_1) \xrightarrow{e} (s_1,c_3) \xrightarrow{b} (s_2,c_3)$ and d to $(s_1,c_2) \xrightarrow{f} (s_1,c_4) \xrightarrow{c} (s_3,c_4)$

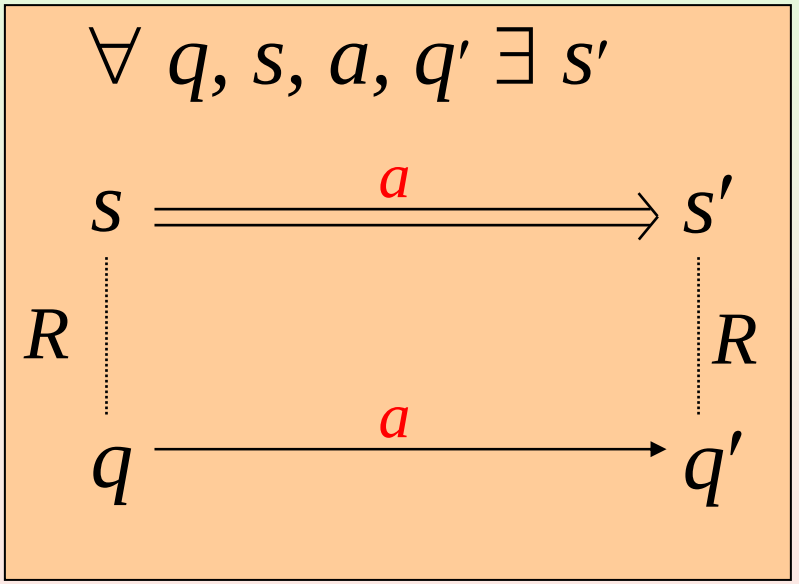
How to Get Compositionality

- Restrict the power of composition
  - Probabilistic reactive modules [AHJ01]
  - Switched probabilistic I/O automata [CLSV04]
- Trace Distribution Precongruence
  - Coarsest precongruence included in preorder
  - Alternative characterizations
    - Principal context [Seg95]
    - Testing [Seg96]
    - Forward simulations [LSV03]

Forward Simulations (Automata)

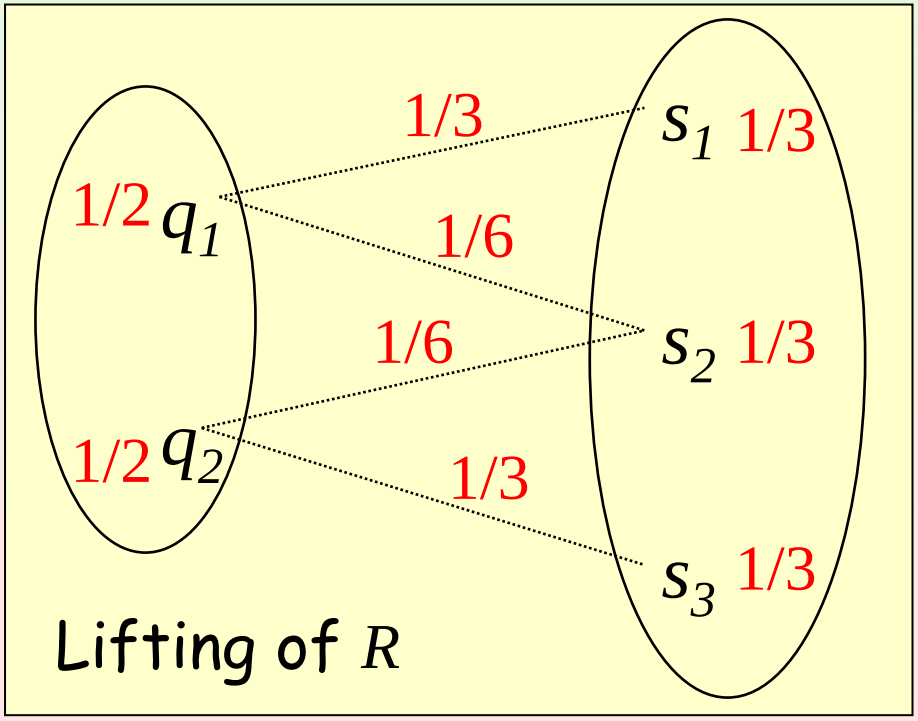
Forward simulation from A_1 to A_2 ($A_1 \leq_F A_2$)
Relation $R \subseteq Q_1 \text{ x } Q_2$ such that

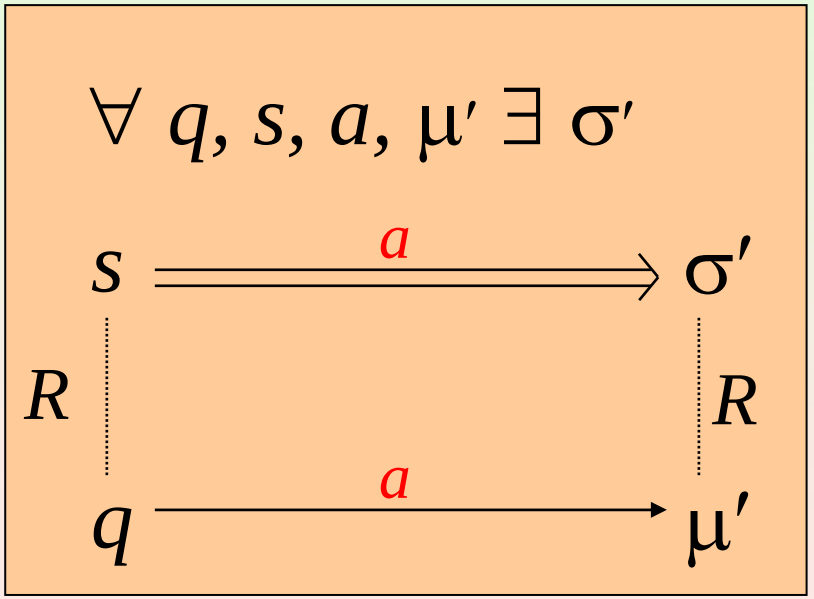




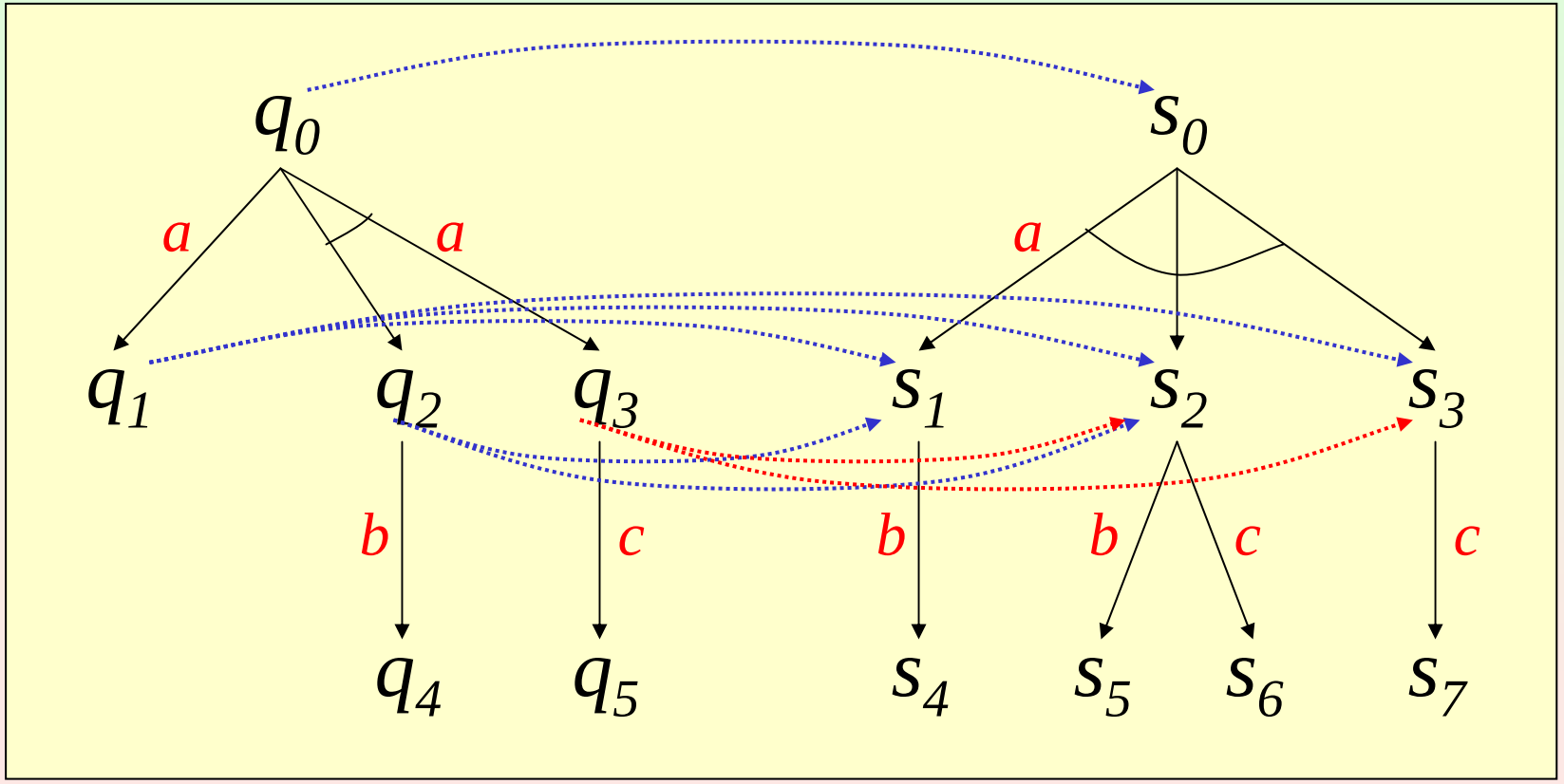
Forward Simulations

Forward simulation from A_1 to A_2 ($A_1 \leq_F A_2$)
Relation $R \subseteq Q_1$ x Q_2 such that





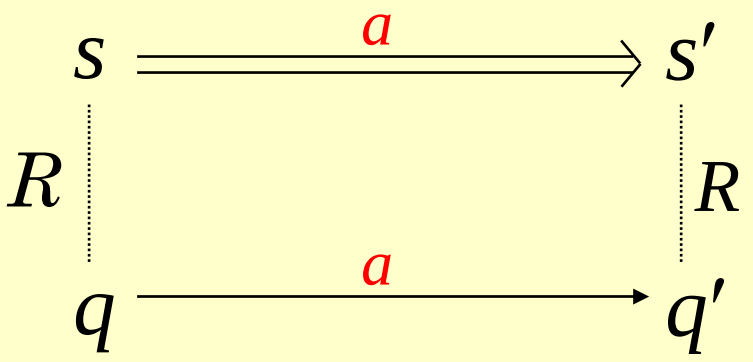
Example: Simulations



Characterization: Forward Simulations (Automata)

Forward simulation from A_1 to A_2 ($A_1 \leq_F A_2$)
Relation $R \subseteq Q_1 \times Q_2$ such that

$\forall\ q, s, a, q'\ \exists\ s'$

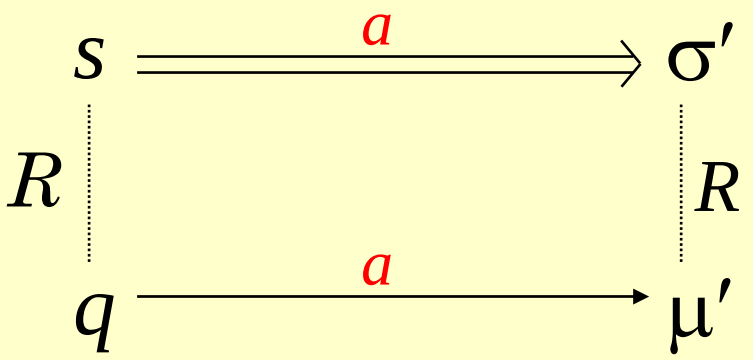


Theorem [LSV02] $A_1 \leq_F A_2$ iff $A_1 \leq_{TDC} A_2$

Characterization:Forward Simulations

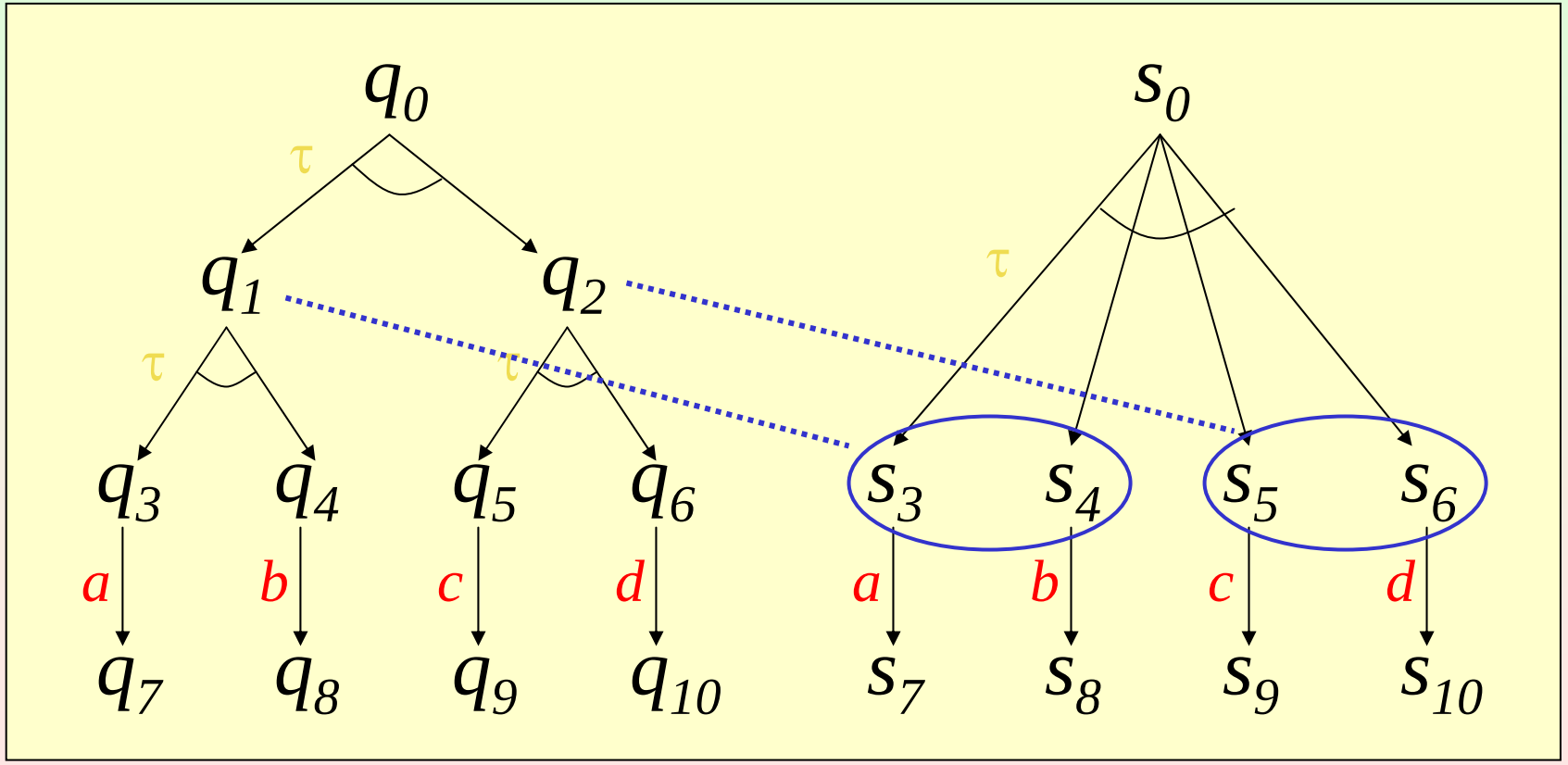
Forward simulation from A_1 to A_2 ($A_1 \leq_F A_2$)
Relation $R \subseteq Q_1 \times Q_2$ such that

$\forall\ q, s, a, \mu'\ \exists\ \sigma'$



Theorem $A_1 \leq_F A_2$ implies $A_1 \leq_{TDC} A_2$

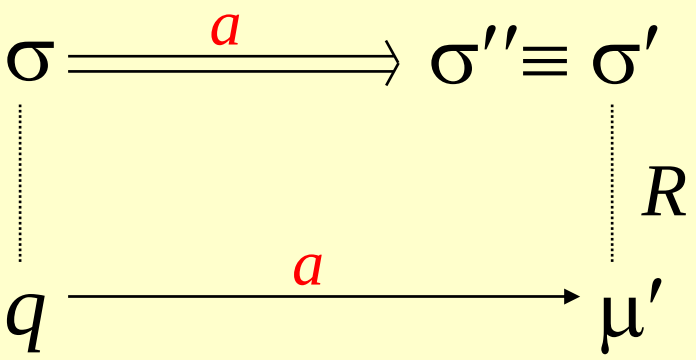
Example: Failure of Weak Forward Simulations



Characterization: Probabilistic Forward Simulations

Forward simulation from A_1 to A_2 ($A_1 \leq_{PF} A_2$)
Relation $R \subseteq Q_1 \text{ x } Disc(Q_2)$ such that

$\forall\ q, \sigma, a, \mu' \ \exists\ \sigma'', \sigma'$ $\quad R$



Theorem [LSV02] $A_1 \leq_{PF} A_2$ iff $A_1 \leq_{TDC} A_2$

Other Important Topics

- Axiomatizations
- Logical characterizations
- Probabilistic language inclusion
- Simulation relations
- Metrics
- Testing
- Model checking
- Timed models
  - Timed Probabilistic Automata
  - Stochastic Transition Systems